Latin hypercubes realizing integer partitions

Tara Kemp Supervisors: Diane Donovan, James Lefevre

School of Mathematics and Physics, The University of Queensland

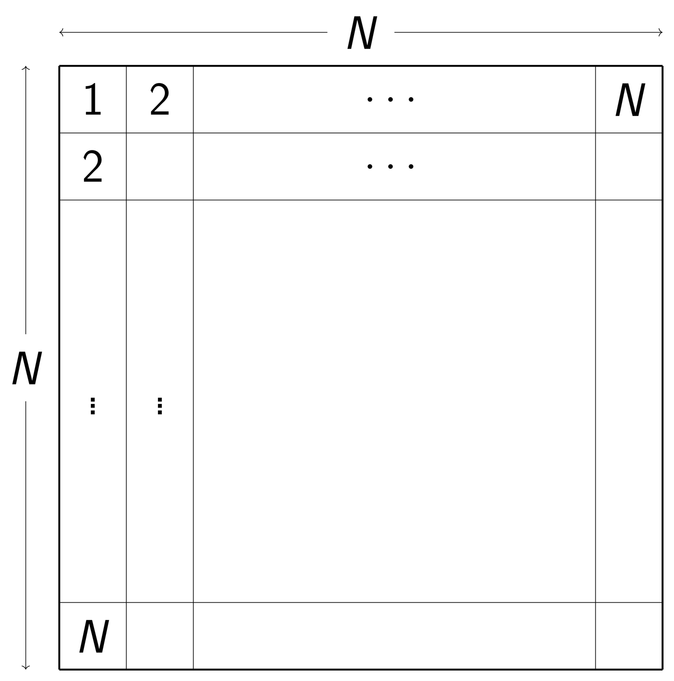
Latin Squares

Have you played Sudoku?

Sudoku grids are an example of a latin square. The aim of Sudoku is to put every number exactly once in every row, in every column, and every 3x3 block.

1	2	3	7	8	9	4	5	6
4								
5								
6			1	2	3			
2			4	6	5			
3			8	9	7			
7								
8								
9								

If you remove subdivision into 3x3 blocks, then you have a latin square!
Latin squares can also be of any size; not just 9x9.



Subsquares

What if we have a square within a square?

If there is a smaller latin square within a latin square, we call the smaller grid a subsquare.

For any partition, a 2-realization is a latin square with a collection of disjoint subsquares corresponding to the parts of the partition. These subsquares must be row, column and symbol disjoint.
The latin squares on the right are 2-realizations with subsquares of order 1 and 2.

1	2	3	4	5	6
2	3	1	5	6	4
3	1	2	6	4	5
6	4	5	1	2	3
4	5	6	2	3	1
5	6	4	3	1	2

A subsquare of order 3

1	2	5	6	3	4
2	1	6	5	4	3
5	6	3	4	1	2
6	5	4	3	2	1
3	4	1	2	5	6
4	3	2	1	6	5

A realization of $(2,2,2)$

1	2	4	5	3
2	1	5	3	4
4	5	3	2	1
5	3	1	4	2
3	4	2	1	5

A realization of $(2,1,1,1)$

The Problem

Which realizations exist?

It is not always possible to find a 2-realization for a partition. The latin square below shows that there is no 2-realization for (1,1), since the subsquares cannot be symbol-disjoint.
This problem of existence is only partially solved.

1	2
2	1

There is no 2-RP(1^2)

Higher Dimensions

Does this work with cubes?

In an m-dimensional latin hypercube, an m-realization is a collection of smaller m-dimensional hypercubes.

The latin squares below show the layers of a 3-RP($2^2 1^1$).

1	2	3	5	4
2	1	4	3	5
3	4	5	1	2
4	5	1	2	3
5	3	2	4	1

2	1	4	3	5
1	2	5	4	3
5	3	1	2	4
3	4	2	5	1
4	5	3	1	2

4	3	5	2	1
3	5	2	1	4
1	2	3	4	5
5	1	4	3	2
2	4	1	5	3

5	4	2	1	3
4	3	1	5	2
2	5	4	3	1
1	2	3	4	5
3	1	5	2	4

3	5	1	4	2
5	4	3	2	1
4	1	2	5	3
2	3	5	1	4
1	2	4	3	5

A 3-realization of $(2,2,1)$

Realizations of Hypercubes

How does the problem change?

There are partitions for which a 3-realization exists, but a 2-realization doesn't.

1	2		2	1
2	1		1	2

3-RP(1^2)

However, if a 2-realization exists for a partition, then a 3-realization does too.
In fact, if a 2-realization exists, then an m-realization exists!
This doesn't work for 3-realizations however, since a 4-realization of (1,1) doesn't exist, as shown below.

1	2		2	1	⋮	2	1		1	2
2	1		1	2	⋮	1	2		2	1

4-RP(1^2)

Solving the Problem

Can't a computer find them?

Unfortunately, the number of possible latin squares or latin cubes gets large very quickly.

Each new dimension also adds another layer of complexity that needs to be considered.

Also, many of the construction methods used for latin squares do not work for latin cubes due to this new complexity.

Results

What have we found for latin cubes?

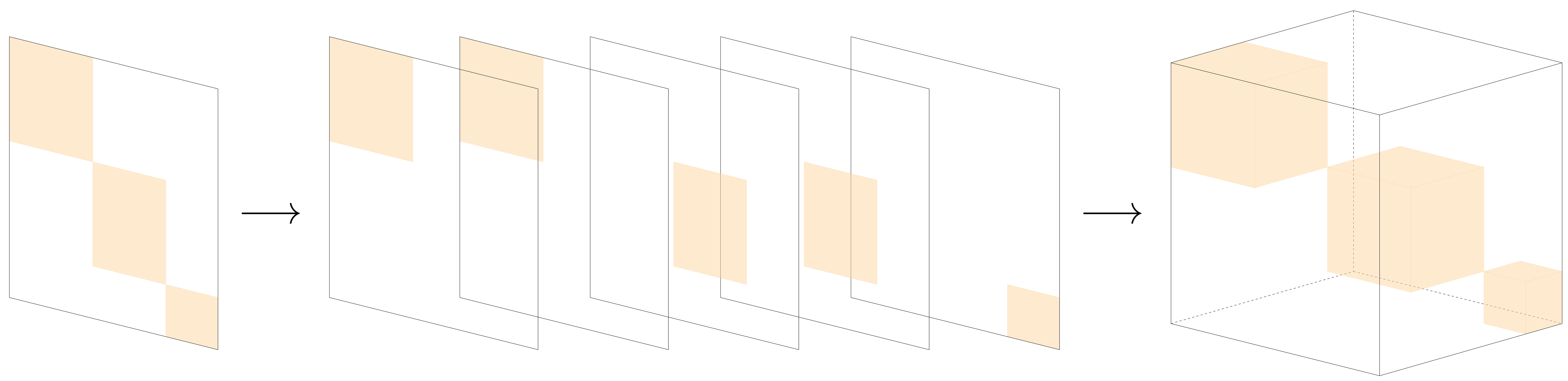
- We have shown that the existence of a latin square for a partition implies the existence of a latin cube. The idea of the construction is shown in the figure below.
- Found necessary (but not sufficient) conditions for the existence of 3-realizations.
- Found new construction methods for 3-realizations.

- There are connections to higher dimensions.
  In general, if an m-realization exists, then a k-realization exists for all $k \equiv 1 \pmod{m-1}$.
  So a latin square implies existence in every higher dimension and a latin cube implies existence in all odd dimensions.
- We determined the existence of 3-realizations for some small patterns of subcubes, where the 2-realization does not exist. The table below shows the bounds in which the realizations exist for these partitions.

	Existence Bounds	
Partition	2-RP	3-RP
a^n	$n \neq 2$ [1]	No bound
$a^1 b^2$	DNE [2]	$a \leq 2b$
$a^1 b^{n-1}$	$a \leq (n-2)b$ [2]	$a \leq (n-1)b$
$a^2 b^1$	DNE [2]	$a \leq 2b*$
$a^2 b^2$	DNE [2]	$a \leq 4b*$
$a^2 b^{n-2}$	$a \leq (n-2)b$ [2,3]	$a \leq 2(n-2)b*$
$a^u b^{n-u}$	No bound [2,3]	No bound

Table: Existence of realizations with at most two distinct parts

*Bound has been proven, but not yet existence in all cases.



[1] Dénes, Pásztor, 1963. [2] Heinrich, 1982. [3] Kuhl, Schroeder, 2018.