Latin squares with five disjoint subsquares

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Latin Squares

Can you fill this grid?

Try to fill in this grid so that each number from 1 to 7 occurs exactly once in each row and column.

1	2					
2	1					
		3	4			
		4	3			
				5	6	
				6	5	
						7

You have filled what is called a latin square!

This latin square is extra interesting because it contains a set of smaller latin squares. These smaller squares are called subsquares, and they are row, column and symbol disjoint.
The subsquares are of orders 2, 2, 2 and 1, so we call this a **realization** of $(2, 2, 2, 1)$.

The Problem

Which realizations exist?

It is not always possible to find a realization for a partition. The latin square below shows that there is no 2-realization for (1,1), since the subsquares cannot be symbol-disjoint.
This problem of existence is only partially solved.

The latin squares on the right are realizations with subsquares of order 1 and 2.

1	2
2	1

There is no 2-RP(1^2)

1	2	5	6	3	4
2	1	6	5	4	3
5	6	3	4	1	2
6	5	4	3	2	1
3	4	1	2	5	6
4	3	2	1	6	5

A realization of $(2, 2, 2)$

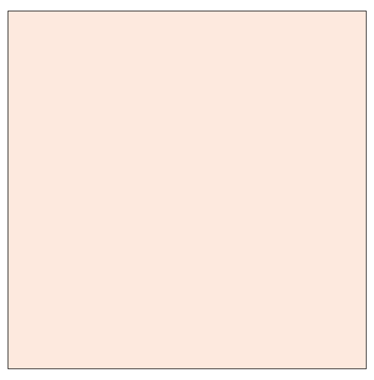
1	2	4	5	3
2	1	5	3	4
4	5	3	2	1
5	3	1	4	2
3	4	2	1	5

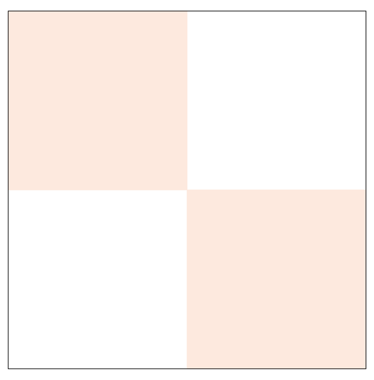
A realization of $(2, 1, 1, 1)$

Limiting Subsquares

What if we specify the number of subsquares?

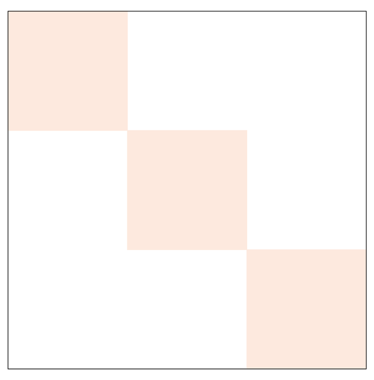
Most of the known results are for partitions with at most four subsquares [1].

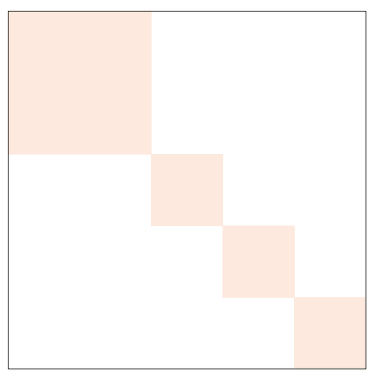




1: The subsquare is the whole square.

2: Regardless of the size of the subsquares, no realization exists.





3: They must all be the same size.

4: At least three of them must be the same size.

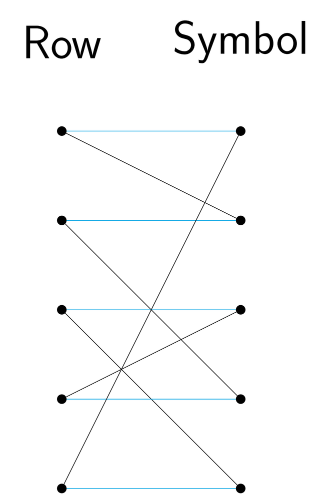
Wide Columns

How can we make latin squares easier to find?

Imagine that you have a column which has two copies of every symbol, rather than just one. You might say this has a "width" of two.
Using a colouring of a bipartite graph, we can actually turn that column into two columns, with each number occurring once in each [2].

1	2
2	4
3	5
3	4
1	5

⇒



⇒

1	2
2	4
3	5
4	3
5	1

Outline Squares

A general method for filling latin squares

Specifying the contents of every cell can be quite difficult, and this makes constructing realizations for a broad range of partitions quite complicated.

If there are k subsquares, then using "wide" columns and rows, we just need to construct an outline square: a $k \times k$ grid, with multiple entries per cell.

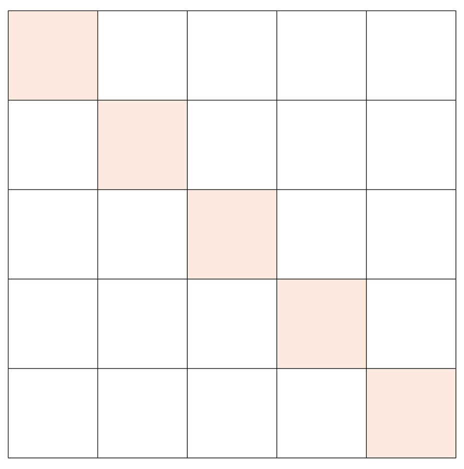
1	1	5	5	3	3
2	2	6	6	4	4
5	5	3	3	1	1
6	6	4	4	2	2
3	3	1	1	5	5
4	4	2	2	6	6

An outline square for $(2, 2, 2)$

Five Subsquares

Which realizations exist?

Using outline squares, we determined exactly which realizations with five subsquares exist. It's a lot more complicated than with four or fewer!



Interesting Results

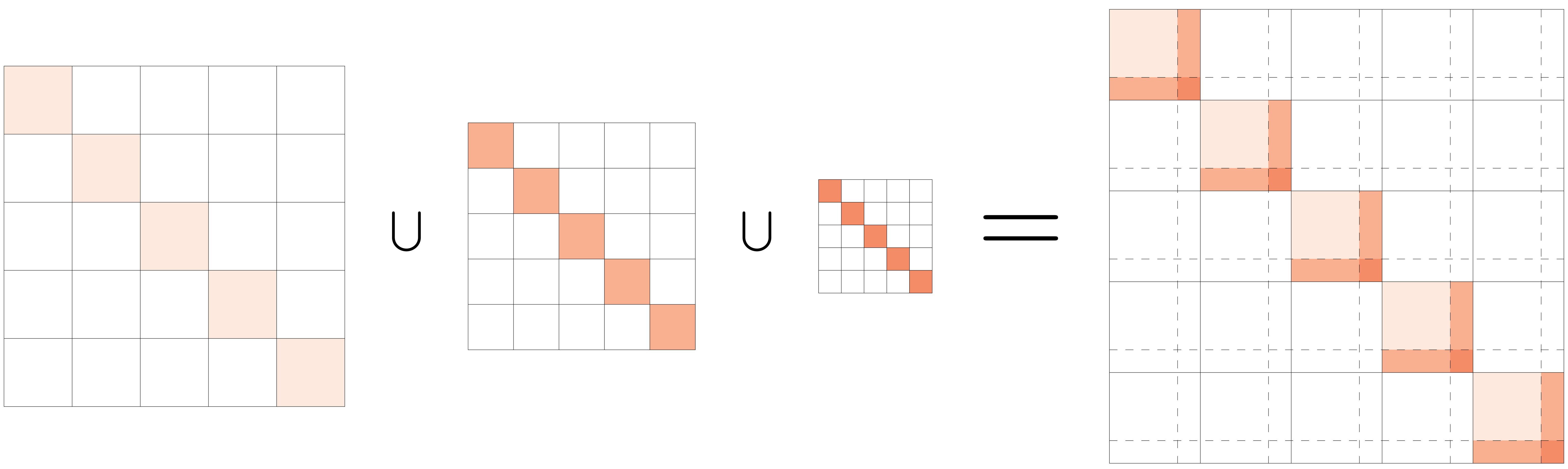
As well as completing the case of five subsquares, we have shown that:

If a realization of $(h_1, h_2, h_3, h_4, h_5)$ exists, then a realization of

$$(h_1 + q, h_2 + q, h_3 + q, h_4 + q, h_5 + q)$$

exists for any positive integer q.

For any realization with at least five subsquares, if the order of the largest subsquare is at most three times the smallest, then the realization exists.



[1] Heinrich, 1976. [2] Hilton, 1980.